Définition 1

Soit $f \in \mathcal{L}(E)$. $\forall u, v \, \langle f(u), f(v) \rangle = \langle u, v \rangle$

Théorème 1

$f \in \mathcal{L}(E)$ et $\dim E = n < +\infty$ Il suffit de montrer l'injectivité

$$\operatorname{Ker} f = \{x \in E, f(x) = 0_E\}$$

Soit $x \in \operatorname{Ker} f$. On a $\|x\|^2 = \langle x, x \rangle = \langle f(x), f(x) \rangle = \|f(x)\|^2 = 0$ donc par séparation:

$$x = O_E$$

Finito.

$$O(E) \subset GL(E)$$

Démonstration

1.

$$s_F^{\perp} : \begin{cases} E = F \oplus F^{\perp} & \to E \\ x = x_F + x_{F^{\perp}} & \mapsto x_F - x_{F^{\perp}} \end{cases}$$

Soient $x, y \in E$.

$$\begin{cases} x & = \underbrace{x_F}_{\in F} + \underbrace{x_{F^{\perp}}}_{\in F^{\perp}} \\ y & = \underbrace{y_F}_{\in F} + \underbrace{y_{F^{\perp}}}_{\in F^{\perp}} \end{cases}$$

$$\begin{aligned} \left\langle s_F^{\perp}(x), s_F^{\perp}(y) \right\rangle &= \langle x_F - x_{F^{\perp}}, y_F - y_{F^{\perp}} \rangle \\ &= \langle x_F, y_F \rangle - \underbrace{\langle x_F, y_{F^{\perp}} \rangle}_{0} - \underbrace{x_{F^{\perp}}, y_F}_{0} + \langle x_{F^{\perp}}, y_{F^{\perp}} \rangle \\ &= \langle x_F, y_F \rangle + \langle x_{F^{\perp}}, y_{F^{\perp}} \rangle \\ &= \langle x, y \rangle \end{aligned}$$

2. $p_F^{\perp}$ est un automorphisme $\iff \operatorname{Ker} p_F^{\perp} = \{0_E\} \iff p_F^{\perp} = p_F^{\{}0_E\} = \mathrm{id}_E$

Démonstration

1. On suppose $f(F) \subset F$