$\boxed{-2}$

$$\mathrm{Ker}(f_{|G}) = \{x \in G,\ f(x) = 0\}$$

$$\mathrm{Ker} f \cap G = \{x \in E,\ f(x) = 0\} \cap G$$

$$= \{x \in E,\ x \in G \text{ et } f(x) = 0\}$$

$$= \{x \in G,\ f(x)\} \qquad \text{car } G \subset E$$

$$= \mathrm{Ker}(f_{|G})$$

$\boxed{-1}$ Soit $f \in \mathcal{L}(E,F)$ et $g \in \mathcal{L}(F,G)$

$$\operatorname{Im} f = \{ f(x),\ x \in E \}$$
$$\operatorname{Ker} g = \{ y \in F,\ g(y) = 0 \}$$

$$\operatorname{Im} f \subset \operatorname{Ker} g \iff \forall y \in \operatorname{Im} f,\ g(y) = 0$$
$$\iff \forall x \in E,\ g(f(x)) = 0$$
$$\iff g \circ f = 0$$

Ex M AL

$\boxed{2}$

Meth 1 $f\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\3\\0\end{pmatrix}$; $f\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}0\\3\\3\end{pmatrix}$

$f \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$

Donc il existe une matrice 3,2 tq

$$M\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\3\\0\end{pmatrix} \text{ et } M\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}0\\3\\3\end{pmatrix}$$

Notons $M=\begin{pmatrix}a & b\\ c & d\\ e & f\end{pmatrix}$

$$M\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}a+2b\\c+2d\\e+2f\end{pmatrix}=\begin{pmatrix}3\\3\\0\end{pmatrix}$$

$$M\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2a+b\\2c+d\\2e+f\end{pmatrix}=\begin{pmatrix}0\\3\\3\end{pmatrix}$$

$$\begin{cases}a+2b=3\\2a+b=0\end{cases} \Rightarrow \begin{cases}a=-1\\b=2\end{cases}$$

$$\begin{cases}c+2d=3\\2c+d=3\end{cases} \Rightarrow \begin{cases}c=1\\d=1\end{cases}$$

$$\begin{cases}e+2f=0\\2e+f=3\end{cases} \Rightarrow \begin{cases}e=2\\f=-1\end{cases}$$

Donc $M=\begin{pmatrix}-1 & 2\\ 1 & 1\\ 2 & -1\end{pmatrix}$ Ainsi $\begin{cases}\mathbb{R}^2 \longrightarrow \mathbb{R}^3\\ \begin{pmatrix}a\\b\end{pmatrix} \mapsto \begin{pmatrix}-a+2b\\a+b\\2a-b\end{pmatrix}\end{cases}$

Meth 2 $\left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right)$ est une base de $\mathbb{R}^3$

car 2 vecteurs non //

et $\left(\begin{pmatrix}3\\3\\0\end{pmatrix}, \begin{pmatrix}0\\3\\3\end{pmatrix}\right)$ est une famille de 2 vecteurs de $\mathbb{R}^3$

donc d'après le cours il existe une unique solution f

Soit $\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$

Décomposons-le dans la base $\mathcal{B} = \left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right)$

ie trouvons λ, μ tels que

$$\begin{pmatrix}x\\y\end{pmatrix} = \lambda\begin{pmatrix}1\\2\end{pmatrix} + \mu\begin{pmatrix}2\\1\end{pmatrix}$$

$$\Leftrightarrow \begin{cases}\lambda + 2\mu = x\\ 2\lambda + \mu = y\end{cases}$$

$$\Leftrightarrow \begin{cases}\lambda = \dfrac{2y-x}{3}\\ \mu = \dfrac{2x-y}{3}\end{cases}$$

et $$f\begin{pmatrix}x\\y\end{pmatrix} = f\left(\frac{2y-x}{3}\begin{pmatrix}1\\2\end{pmatrix} + \frac{2x-y}{3}\begin{pmatrix}2\\1\end{pmatrix}\right)$$

$$= \frac{2y-x}{3}\begin{pmatrix}3\\3\\0\end{pmatrix} + \frac{2x-y}{3}\begin{pmatrix}0\\3\\3\end{pmatrix}$$

$$= \begin{pmatrix}2y-x\\2y-x\\0\end{pmatrix} + \begin{pmatrix}0\\2x-y\\2x-y\end{pmatrix} = \begin{pmatrix}2y-x\\x+y\\2x-y\end{pmatrix}$$

$\boxed{3}$ $\overset{\text{déf}}{\exists}\ \lambda \in \mathbb{K},\ \phi = \lambda \mathrm{Tr}.$

Pour $i \neq j$, $E_{i,j} = E_{i,j} \times E_{j,j}$ et $E_{j,j} \times E_{i,j} = (0)_n$

car, en général : $\forall i,j,k,l \in \mathbb{N}$, $E_{i,j} \times E_{k,l} = E_{i,j} \times \begin{array}{cc} & \begin{array}{ccc} & l & \end{array} \\ & \begin{pmatrix} 0 & \cdots\ 0\ \cdots & 0 \\ \vdots & \vdots & \vdots \\ & 0 & \\ & 1 & \\ & 0 & \\ \vdots & \vdots & \vdots \\ 0 & \cdots\ 0\ \cdots & 0 \end{pmatrix} \begin{array}{c} \\ \\ \\ \leftarrow k \\ \\ \\ \\ \end{array} \end{array}$

$= k$-ième colonne de $E_{i,j}$

Donc, $\begin{cases} i \neq j \Rightarrow E_{i,j} \times E_{k,l} = (0)_n & \text{pour } j \neq k \\ i = j \Rightarrow E_{i,j} \times E_{k,l} = E_{i,l} \end{cases}$

donc $E_{i,j} \times E_{k,l} = \delta_{j,k} E_{i,l}$

$$\begin{aligned} \phi(E_{i,j}) &= \phi(E_{i,j} \times E_{j,j}) \\ &= \phi(E_{j,j} \times E_{i,j}) \\ &= \phi((0)_n) \\ &= 0 \end{aligned}$$

Soit $A \in \mathcal{M}_n(\mathbb{K})$

$$\begin{aligned} \phi(A) &= \phi\Big(\sum a_{ij} E_{ij}\Big) \\ &= \phi\Big(\sum_{i=j} a_{ii} E_{ii} + \sum_{i \neq j} a_{ij} E_{ij}\Big) \\ &= \sum_{i=j} a_{ii}\, \phi(E_{ii}) + \sum_{i \neq j} a_{ij}\, \phi(E_{ij}) \end{aligned}$$

$= \sum_i a_{ii} \phi(E_{ii}) + 0$

$= \sum_i a_{ii} \lambda$ avec $\lambda = \phi(E_{ii})$ pour tout i

$\boxed{4}$ On sait que $S_n(\mathbb{K}) \oplus A_n(\mathbb{K}) = \mathcal{M}_n(\mathbb{K})$

Notons $\varphi_1 : \begin{cases} S_n(\mathbb{K}) \longrightarrow \mathcal{M}_n(\mathbb{K}) \\ S \longmapsto 3S \end{cases}$

$\varphi_2 : \begin{cases} A_n(\mathbb{K}) \longrightarrow \mathcal{M}_n(\mathbb{K}) \\ A \longmapsto 2A \end{cases}$

D'après la prop généralisée, il existe une unique application

$$\varphi \text{ tq } \begin{cases} \varphi_{|S_n(\mathbb{K})} = \varphi_1 \\ \varphi_{|A_n(\mathbb{K})} = \varphi_2 \end{cases}$$

Soit $M \in \mathcal{M}_n(\mathbb{K})$

On a $\begin{cases} M = A + S \\ {}^tM = {}^t(A+S) \end{cases}$ avec $\begin{cases} A \in A_n(\mathbb{K}) \\ S \in S_n(\mathbb{K}) \end{cases}$

$= {}^tA + {}^tS$ par linéarité de t.

$= -A + S$

On a $\begin{cases} S = \dfrac{M + {}^tM}{2} \\ A = \dfrac{M - {}^tM}{2} \end{cases}$ donc $\varphi : \begin{cases} \mathcal{M}_n(\mathbb{K}) \longrightarrow \mathcal{M}_n(\mathbb{K}) \\ M \longmapsto \varphi_1(S) + \varphi_2(A) \end{cases}$

$= 3S + A$

$= \frac{1}{2}(5M + {}^tM$

6/g

$$\operatorname{Ker} g = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}, \begin{cases} x - y + z = 0 \\ -2x - 4z + 4t = 0 \\ y + z - 2t = 0 \\ 3x - 2y + 4z - 2t = 0 \end{cases} \right\}$$

Résolvons $\begin{cases} x - y - z = 0 \\ -2y - 2z + 4t = 0 \\ y + z - 2t = 0 \\ y + z - 2t = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} x = y - z \\ y = y \\ z = z \\ t = \frac{y}{2} + \frac{z}{2} \end{cases}$$

$$\Leftrightarrow \operatorname{Ker} g = \operatorname{Vect}\Bigg(\underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1/2 \end{pmatrix}}_{\text{non } //} \Bigg)$$

$$\operatorname{Im} g = \left\{ g\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{K}^4 \right\}$$

$$= \left\{ x\begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix} + y\begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix} + z\begin{pmatrix} 1 \\ -4 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 0 \\ 4 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{K}^4 \right\}$$

$$= \operatorname{Vect}\left(\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -2 \\ -2 \end{pmatrix} \right)$$

Or $\begin{pmatrix} 1 \\ -4 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ -2 \\ -2 \end{pmatrix}$

Donc $\operatorname{Im} g = \operatorname{Vect}\left(\begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -2 \\ -2 \end{pmatrix}\right)$

Or $\begin{pmatrix} 0 \\ 4 \\ -2 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}$

Donc $\operatorname{Im} g = \operatorname{Vect}\left(\underbrace{\begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \end{pmatrix}}_{\text{non } //}\right)$

$\boxed{8}$

Meth 1) Soit $f \in \mathcal{L}(E, \mathbb{K}) \setminus \{0_{\mathcal{L}(E,\mathbb{K})}\}$

$$\mathrm{Im}(f) = \{f(x),\ x \in E\}$$

Or $f \neq 0$ donc il existe $x \in E$ tel que

$$f(x) = u \neq 0$$

Donc $u \in \mathrm{Im} f$

Par linéarité, $\forall \lambda \in \mathbb{K},\ f(\lambda x) = \lambda u$

Donc $\mathrm{Im} f \supset \{\lambda u,\ \lambda \in \mathbb{K}\} = \mathbb{K}$

Et par hypothèse, $\mathrm{Im} f \subset \mathbb{K}$

Donc $\mathrm{Im} f = \mathbb{K}$

Meth 2

On sait que

- $\mathrm{Im} f$ sev $\mathbb{K}$
- Les sevs de $\mathbb{K}$ sont $\{0\}$ et $\mathbb{K}$

Or $f \neq 0$ donc $\mathrm{Im} f \neq \{0\}$ donc $\mathrm{Im} f = \mathbb{K}$

$\boxed{9/1}$

Mq $\phi \in FL \setminus \{0\}$.

On a ϕ

- linéaire car eval est linéaire
- est une FL car $\begin{cases} \text{src } \phi = E \\ \text{but } \phi = \mathbb{K} \end{cases}$
- non nulle car $\phi \in \mathbb{K}_0[X]$

$\boxed{9/2}$

$$\text{Ker } \phi = (X-a)\mathbb{K}[X]$$

Une base : la base de Taylor

$$((X-a), (X-a)^2, \dots)$$

EXMAL $\boxed{10}$

$f, g \in E^* \setminus \{0_{E^*}\}$

$\underbrace{\mathrm{Ker}\, f = \mathrm{Ker}\, g}_{\text{hyperplans}} \iff f \mathbin{/\!/} g$

$\boxed{\Leftarrow}$ Supp $f \mathbin{/\!/} g$, ie $\overset{\text{def}}{\exists} \lambda \in \mathbb{K}^*$ tel que $f = \lambda g$

$$\begin{aligned} \mathrm{Ker}\, f &= \{x \in E,\ f(x) = 0\} \\ &= \{x \in E,\ \lambda g(x) = 0\} \\ &= \{x \in E,\ g(x) = 0\} \quad \text{car } \lambda \neq 0 \\ &= \mathrm{Ker}\, g \end{aligned}$$

$\boxed{\Rightarrow}$ Supposons $\mathrm{Ker}\, f = \mathrm{Ker}\, g$

$\mathrm{Ker}\, g$ est un hyperplan donc il existe $u \in E \setminus \{0_E\}$ tel que $\mathrm{Ker}\, g \oplus \mathrm{Vect}\, u = E$ avec $g(u) \neq 0$.

$\mathrm{Ker}\, f = \mathrm{Ker}\, g$ donc $\mathrm{Ker}\, f \oplus \mathrm{Vect}\, u = E$

$$g : \begin{cases} E = \operatorname{Ker} g \oplus \operatorname{Vect} u \longrightarrow \mathbb{K} \\ x = x_K + \lambda u \longmapsto \lambda g(u) \end{cases}$$

$$f : \begin{cases} E = \overbrace{\operatorname{Ker} g}^{\operatorname{Ker} f} \oplus \operatorname{Vect} u \longrightarrow \mathbb{K} \\ x = x_K + \lambda u \longmapsto \lambda f(u) \end{cases}$$

Trouvons μ tq $f = \mu g$.

Posons $\mu = \dfrac{f(u)}{g(u)}$

$$\begin{aligned} \mu g &= x \mapsto \mu \lambda g(u) \\ &= x \mapsto \lambda f(u) \\ &= f \end{aligned}$$

EX$^{\underline{o}}$ AL

$\boxed{11}$

$$\mathrm{rg}(f+g) = \dim \mathrm{Im}\,(f+g)$$

$\mathrm{Im}\,(f+g) \subset \mathrm{Im} f + \mathrm{Im}\, g$ (et en général on ne peut pas dire mieux)

$$\mathrm{Im}(f+g) = \{ f(x)+g(x), \quad x \in E \}$$

$$\mathrm{Im}\, f + \mathrm{Im}\, g = \{ f(x), x \in E \} + \{ g(x), x \in E \}$$

$$= \{ f(x)+g(y), \ (x,y) \in E^2 \}$$

d'où l'inclusion (en prenant "$(x,y) = (x,x)$")

On a $\dim E \leq c$:

$$\mathrm{rg}(f+g) = \dim \mathrm{Im}\,(f+g) \leq \dim(\mathrm{Im}\, f + \mathrm{Im}\, g)$$

$$= \dim \mathrm{Im} f + \dim \mathrm{Im}\, g \underbrace{- \dim(\mathrm{Im}\, f \cap \mathrm{Im}\, g)}_{\in \mathbb{N}}$$ d'après Grassman

$$\leq \dim \operatorname{Im} f + \dim \operatorname{Im} g$$

$$= \operatorname{rg} f + \operatorname{rg} g$$

$\boxed{12}$

AB inversible $\Leftrightarrow$ $(AB)\times id \in \mathcal{L}(\mathbb{K}^p)$ bijective

BA inversible $\Leftrightarrow$ $(BA)\times id \in \mathcal{L}(\mathbb{K}^q)$ bijective

$(AB)\times id = (A\times id)\circ(B\times id) \in$ donc $\begin{cases} A\,id \in \\ B\,id \in \end{cases}$

$(BA)\times id = (B\times id)\circ(A\times id) \in$ donc $\begin{cases} A\,id \in \\ B\,id \in \end{cases}$

Donc $(A\,id)$ est un isomorphisme

donc $\dim \mathbb{K}^p = \dim \mathbb{K}^q$

donc $p = q$

14/1/a

$$\operatorname{Ker} g \cap \operatorname{Ker} h = \left\{ x \in E, \begin{cases} x + f(x) + \dots + f^{p-1}(x) = 0_E \\ f(x) = x \end{cases} \right\}$$

$$= \left\{ x \in E, \begin{cases} x + x + f(x) + \dots + f^{p-2}(x) = 0_E \\ f(x) = x \end{cases} \right\}$$

$$\vdots$$

$$= \{ x \in E, \ p \cdot x = 0_E \}$$

$$= \{ 0_E \} \quad \text{car } p \neq 0$$

14/1/b

$$\dim \operatorname{Ker} g + \dim \operatorname{Ker} h = \dim(\operatorname{Ker} g \cap \operatorname{Ker} h) + \dim(\operatorname{Ker} g + \operatorname{Ker} h)$$

d'après Grassman

$$= \dim(\operatorname{Ker} g + \operatorname{Ker} h)$$

$$\leq \dim(g + h)$$

$$\leq n$$

$\boxed{14/2/a}$

$\mathrm{Im}\, g = \{g(x),\ x \in E\}$

$\mathrm{Ker}\, h = \{x \in E,\ f(x) = x\}$

Soit $y \in \mathrm{Im}\, g$. $\overset{\text{def}}{\exists} x \in E,\ y = g(x)$.

$$f(y) = f(y(x)) = f(x + f(x) + f^2(x) + \dots + f^{p-1}(x))$$

Or $f(y) = f(x) + f^2(x) + \dots + f^p(x)$

$$= x + f(x) + \dots + f^{p-1}(x)$$

$$= y(x)$$

$$= y$$

donc $y \in \mathrm{Ker}\, h$

donc $\mathrm{Im}\, g \subset \mathrm{Ker}\, h$

$\boxed{14/2/b}$

$\mathrm{Im}\, g \subset \mathrm{Ker}\, h$

donc $\mathrm{rg}\, g \subset \dim \mathrm{Ker}\, h$

$$\dim \mathrm{Ker}\, g + \dim \mathrm{Ker}\, h = \dim \mathrm{Ker}\, h + n - \mathrm{rg}\, g \quad \text{par thm du rg}$$

$$\geqslant \dim \mathrm{Ker}\, h + n - \dim \mathrm{Ker}\, h$$

$$= n$$

14/3

On a $\begin{cases} \dim \operatorname{Ker} h + \dim \operatorname{Ker} g = n \\ \operatorname{Ker} h \cap \operatorname{Ker} g = \{0_E\} \end{cases}$

Donc par caractérisation des supplémentaires en dim N :

$$\operatorname{Ker} g \oplus \operatorname{Ker} h = E$$

EXM AL

$\boxed{17/1}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -y-2z \\ -2x-y-4z \\ x+y+3z \end{pmatrix} = \begin{pmatrix} 0 & -1 & -2 \\ -2 & -1 & -4 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Meth 1

Soit $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{K}^3$.

$$p\left(p\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = p\begin{pmatrix} -y-2z \\ -2x-y-4z \\ x+y+3z \end{pmatrix}$$

$$= \begin{pmatrix} -(2x-y-4z) - 2(x+y+3z) \\ -2(-y-2z) - (-2x-y-4z) - 4(x+y+3z) \\ (-y-2z) + (-2x-y-4z) + 3(x+y+3z) \end{pmatrix}$$

$$= \begin{pmatrix} -y-2z \\ -2x-y-4z \\ x+y+3z \end{pmatrix}$$

$$= p\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Meth 2

$$p \circ p = \begin{pmatrix} 0 & -1 & -2 \\ -2 & -1 & -4 \\ 1 & 1 & 3 \end{pmatrix}^2 \mathrm{id}_{\mathbb{K}^3}$$

$$= \begin{pmatrix} 0 & -1 & -2 \\ -2 & -1 & -4 \\ 1 & 1 & 3 \end{pmatrix} \mathrm{id}_{\mathbb{K}^3}$$

$$= p$$

$\boxed{17/2}$

$$\operatorname{Im} p = \left\{ p\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{K}^3 \right\}$$

$$= \operatorname{Vect}\left(\underbrace{\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}}_{u}, \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}_{v}, \underbrace{\begin{pmatrix} -2 \\ -4 \\ -3 \end{pmatrix}}_{u+v} \right)$$

$$= \operatorname{Vect}\left(\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right) \quad \text{par oxydoréduction}$$

$$\operatorname{Ker} p = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{K}^3, \ p\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{K}^3, \ \begin{cases} -y - 2z = 0 \\ -2x - y - 4z = 0 \\ x + y - 3z = 0 \end{cases} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{K}^3, \ \begin{cases} y = -2z \\ -2x = 2z \\ x = -z \end{cases} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{K}^3, \ \begin{cases} y = -2z \\ x = -z \end{cases} \right\}$$

$$= \operatorname{Vect}\left(\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right)$$

$\boxed{17/2}$ Par A-S.

$$F_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{K}^3,\ x+y+z=0 \right\} ; \quad G_2 = \left\{ \begin{pmatrix} \lambda \\ 0 \\ \lambda \end{pmatrix},\ \lambda \in \mathbb{K} \right\}$$

Soit $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{K}^3$.

<u>Analyse</u> Considérons deux

tel que $\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \lambda \\ 0 \\ \lambda \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

avec $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F_3$ et $\begin{pmatrix} \lambda \\ 0 \\ \lambda \end{pmatrix} \in G_3$.

On a $\begin{cases} a = x + \lambda \\ b = y \\ c = z + \lambda \end{cases}$

On somme les trois

$$a+b+c = \underbrace{x+y+z}_{\in F_3} + 2\lambda$$

On a donc $\begin{cases} \lambda = \dfrac{a+b+c}{2} \\ x = \dfrac{a-b-c}{2} \\ y = b \\ z = \dfrac{c-a-b}{2} \end{cases}$

Synthèse Testons nos candidats

- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in G_2$ car $\dfrac{\alpha+\beta+\gamma}{2} \in \mathbb{K}$

- $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \lambda \\ 0 \\ \lambda \end{pmatrix}$: ok!

- $\begin{pmatrix} \dfrac{\alpha-\beta-\gamma}{2} \\ \dfrac{\gamma-\alpha-\beta}{2} \end{pmatrix} \in F_2$ car $\dfrac{\alpha-\beta-\gamma+\gamma-\beta-\alpha}{2}+\beta$

$= -\dfrac{2\beta}{2} + \beta = 0$

Conclusion ok!

$$p_{F_2}^{G_2}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \dfrac{a-b-c}{2} \\ \dfrac{c-a-b}{2} \end{pmatrix}$$

$\boxed{18/1}$

$$s : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+2y+2z \\ 2x+y+2z \\ -2x-2y-3z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

avec $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$

on a $A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 = id_{\mathcal{M}_3(\mathbb{K})}$

Donc $A^2 = A$

$s \circ s = id$

On a s la symétrie par rapport à $Ker(s-id)$, parallèlement à $Ker(s+id)$

$$Ker(s-id) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{K}^3, \begin{pmatrix} 2y+2z \\ 2x+2z \\ -2x-2y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{K}^3, \begin{cases} x = -z \\ y = -z \end{cases} \right\}$$

$$= Vect\left(\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right)$$

$$Ker(s+id) = Ker \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ -2 & -2 & -2 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{K}^3, x+y+z = 0 \right\}$$

$$= Vect\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$\boxed{18/2}$

$$F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, x = y \right\} = \mathrm{Vect}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$G = \mathrm{Vect}\left(\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \right)$$

<u>Analyse</u> considérons une décomposition convenable de $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{K}^3$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} -a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}}_{\in F} + \underbrace{\begin{pmatrix} c \\ -2c \\ 2c \end{pmatrix}}_{\in G}$$

donc $\begin{cases} a = -2x - y \\ b = -y - x \\ c = -2x - 2y - z \end{cases}$

donc $\begin{cases} x = -a + c \\ y = a - 2c \\ z = b + 2c \end{cases}$

<u>Synthèse</u>

- $\begin{pmatrix} -a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in F$: ok!
- $\begin{pmatrix} c \\ -2c \\ 2c \end{pmatrix} \in G$, ok!

$$\begin{cases} a+c = 2x+y-y-x = x \\ a-2c = 2x-y+2y+2z = y \\ b+2c = 2x+2y+z-y-2x = z \end{cases}$$

donc ok!

$$s\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \cdots$$